Q1a

a) If $z = 5/2$ is a root, then (By the Factor Theorem)

$$2z^3 - 13z^2 + 60z - 100 = (2z-5)(z^2+bz+20)$$
$$= 2z^3 + (2b-5)z^2 + (40-5b)z - 100$$

Check that $b=-4$ works here too

Set x^2 coefficients equal:

So $2b - 5 = -13 \Rightarrow b = -4$

The other roots of the equation will be the solutions to

$$z^2 - 4z + 20 = 0$$
$$(z-2)^2 - 4 + 20 = 0$$
$$(z-2)^2 = -16$$
$$z - 2 = \pm\sqrt{-16} = \pm 4i$$
$$z = 2 \pm 4i$$

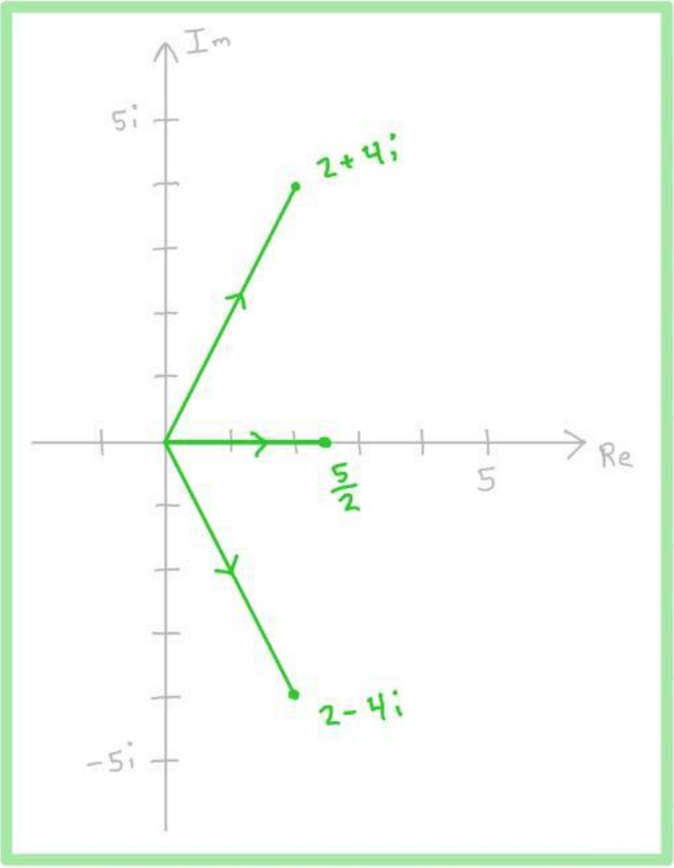
Could also use quadratic formula here

Therefore the roots of the equation are

$$\boxed{\frac{5}{2},\ 2+4i,\ 2-4i}$$

Q1b

b)



Note: You don't need the position vectors to get the marks here – just showing the three points is enough.

Q2

(i) $\frac{1}{z^*} = \frac{1}{x-iy} \times \frac{x+iy}{x+iy} = \frac{x+iy}{x^2+y^2} = \frac{z}{|z|^2}$ $\quad |z| = \sqrt{x^2+y^2}$

$$\boxed{\text{So } \frac{1}{z^*} = \frac{z}{|z|^2}}$$

(ii) From part (i), $\frac{1}{z^*} = \frac{1}{|z|^2} z$.

So $\frac{1}{z^*}$ is z multiplied by a positive real number, therefore $\arg\left(\frac{1}{z^*}\right) = \arg(z)$

Also, $z^* = x - iy$ which is a reflection of z in the real axis. Therefore $\arg(z^*) = -\arg(z)$.

Combining these gives

$$\arg\left(\frac{1}{z^*}\right) = -\arg(z^*)$$

(iii) $z + z^* = (x+iy) + (x-iy) = 2x \in \mathbb{R}$

And $x < 0 \Rightarrow 2x < 0$

$$\boxed{\text{So } \arg(z+z^*) = \pi}$$

For part (ii)

① If you try to show this using $\tan^{-1}\left(\frac{y}{x}\right)$, you have to account for the different cases depending on what quadrant of the Argand diagram z is in.

② If z is a negative real number, then $z = z^*$ and the principal argument of both of them is π. But in the Argand diagram an argument of π or an argument of $-\pi$ are the same thing, so $\arg\left(\frac{1}{z^*}\right) = -\arg(z^*)$ still holds.

Q3A

a) $z_2 = \frac{z_1}{-2+i} \times \frac{-2-i}{-2-i} = \frac{(-2-i)z_1}{5}$

$z_2 = \frac{1}{5}(-2-i)(1+pi)$

$= \frac{1}{5}\left((p-2)+i(-2p-1)\right)$

For positive real number k, $|kz| = k|z|$

$|z_2| = \frac{1}{5}\sqrt{(p-2)^2+(-2p-1)^2} = \sqrt{58}$

So $\frac{1}{25}\left((p-2)^2+(-2p-1)^2\right) = 58$

$p^2 \cancel{-4p} + 4 + 4p^2 \cancel{+4p} + 1 = 25(58)$

$5p^2 + 5 = 25(58)$

$p^2 + 1 = 5(58) = 290$

$p^2 = 289$

$\boxed{p = 17 \text{ or } -17}$

Q3B

From part (a), $p = 17$ or -17

and $z_2 = \frac{1}{5}\left((p-2)+i(-2p-1)\right)$

b) If $p = 17$, then $z_2 = 3 - 7i$ and

$\arg(z_2) = -\tan^{-1}\left(\frac{7}{3}\right) = -1.165904\ldots$

If $p = -17$, then $z_2 = -\frac{19}{5} + \frac{33}{5}i$ and

$\arg(z_2) = \pi - \tan^{-1}\left(\frac{33}{19}\right) = 2.093199\ldots$

So $p = -17$ and $z_2 = -\frac{19}{5} + \frac{33}{5}i$

$z_2^* = -\frac{19}{5} - \frac{33}{5}i$

$$\boxed{\mathrm{Im}(z_2^*) = -\frac{33}{5}}$$

Alternative method without calculating arguments

Note that $\frac{\pi}{2} < 2.09 < \pi$. So $\arg(z_2) = 2.09$ means that z_2 is in the 2^{nd} quadrant on an Argand diagram. Of the two possibilities, only $-\frac{19}{5} + \frac{33}{5}i$ is in the 2^{nd} quadrant.

So $z_2 = -\frac{19}{5} + \frac{33}{5}i$ and $\mathrm{Im}(z_2^*) = -\frac{33}{5}$.

Q4A

a) $r = |z|$ $\theta = \arg z$ with $-\pi < \arg z \leq \pi$

$$|z| = \sqrt{(\sqrt{3}-3)^2 + (\sqrt{3}+3)^2}$$

$$= \sqrt{12 - \cancel{6\sqrt{3}} + 12 + \cancel{6\sqrt{3}}} = \sqrt{24} = 2\sqrt{6}$$

Note that $\sqrt{3} - 3 = -(3-\sqrt{3}) < 0$, so
z is in 3^{rd} quadrant of Argand diagram.

Therefore

$$\arg z = -\pi + \tan^{-1}\left(\frac{3+\sqrt{3}}{3-\sqrt{3}}\right) = -\frac{7\pi}{12}$$

So

$$\boxed{z = 2\sqrt{6}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)}$$

Q4B

b) $|w| = \sqrt{2}\,|z| = \sqrt{2}\,(2\sqrt{6}) = 4\sqrt{3}$

$$\arg(w) = 2(\pi + \arg(z))$$

$$= 2\left(\pi + \left(-\frac{7\pi}{12}\right)\right) = \frac{5\pi}{6}$$

So

$$w = 4\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$= 4\sqrt{3}\left(-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right)$$

$$\boxed{w = -6 + 2\sqrt{3}\,i}$$

Q5

$$z = r(\cos\theta + i\sin\theta)$$

$r = |z|$

$\theta = \arg(z)$

$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$ [Compound angle formula]

$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$ [Compound angle formula]

$$\sin^2 A + \cos^2 A \equiv 1$$

$|z_1| = r_1 \quad \arg(z_1) = \theta_1 \quad |z_2| = r_2 \quad \arg(z_2) = \theta_2$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}\left(\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right)$$

$$= \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$$

Therefore

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} \qquad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

It follows that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Q6a

$z_1 = -4\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)$

$z_2 = 2\left(\cos\frac{2\pi}{3} - \mathrm{i}\sin\frac{2\pi}{3}\right)$

(a) Re-express z_1 and z_2 in correct modulus-argument form with θ in the interval $-\pi < \theta \leq \pi$. [3]

$z = r(\cos\theta + i\sin\theta)$, $r = |z|$, $\theta = \arg(z)$

(b) Work out

(i) $z_1 z_2$

(ii) $\frac{z_2}{z_1}$

giving your answers in modulus-argument form with θ in the interval $-\pi < \theta \leq \pi$. [4]

(c) (i) On an Argand diagram, show the complex numbers z_1, z_2 and $z_1 + z_2$ as position vectors.

(ii) Use your diagram to explain briefly why we speak of a 'parallelogram rule' for complex number addition. [4]

Note: You could also do part (a) by converting z_1 and z_2 into a+bi form and calculating their moduluses and arguments from that.

$\cos(-x) = \cos x$ $\quad$ $\cos(\pi - x) = -\cos x$

$\sin(-x) = -\sin x$ $\quad$ $\sin(\pi - x) = \sin x$

$\cos(-\pi + x) = -\cos x$

$\sin(-\pi + x) = -\sin x$

a) $z_1 = 4\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 4\left(\cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)\right)$

$$\boxed{z_1 = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$$

$z_2 = 2\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

$= 2\left(\cos\left(-\frac{2\pi}{3}\right) - i\left(-\sin\left(-\frac{2\pi}{3}\right)\right)\right)$

$$\boxed{z_2 = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)}$$

Q6B

$z_1 = -4\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right) = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$z_2 = 2\left(\cos\frac{2\pi}{3} - \mathrm{i}\sin\frac{2\pi}{3}\right) = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

(a) Re-express z_1 and z_2 in correct modulus-argument form with θ in the interval $-\pi < \theta \leq \pi$. [3]

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(c) (i) On an Argand diagram, show the complex numbers z_1, z_2 and $z_1 + z_2$ as position vectors.

(ii) Use your diagram to explain briefly why we speak of a 'parallelogram rule' for complex number addition. [4]

$|z_1 z_2| = |z_1||z_2|$ $\quad$ $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $\quad$ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

b) (i) $|z_1 z_2| = (4)(2) = 8$

$\arg(z_1 z_2) = \frac{2\pi}{3} + \left(-\frac{2\pi}{3}\right) = 0$

$$\boxed{z_1 z_2 = 8(\cos 0 + i\sin 0)}$$

(ii) $\left|\frac{z_2}{z_1}\right| = \frac{2}{4} = \frac{1}{2}$

$\arg\left(\frac{z_2}{z_1}\right) = -\frac{2\pi}{3} - \left(\frac{2\pi}{3}\right) = -\frac{4\pi}{3}$

But $-\frac{4\pi}{3} < \pi$ outside $-\pi < \theta \leq \pi$ interval

$\arg\left(\frac{z_2}{z_1}\right) = -\frac{4\pi}{3} + 2\pi = \frac{2\pi}{3}$ this is the principal argument

$$\boxed{\frac{z_2}{z_1} = \frac{1}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$$

Note: It follows from part (i) that $z_1 z_2 = 8$, and from part (ii) that

$\frac{z_2}{z_1} = \frac{z_1}{8} \Rightarrow \frac{z_2}{z_1} = \frac{z_1}{z_1 z_2} = \frac{1}{z_2} \Rightarrow z_1 = z_2^2$

Q6C

$z_1 = -4\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right) = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$z_2 = 2\left(\cos\frac{2\pi}{3} - \mathrm{i}\sin\frac{2\pi}{3}\right) = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

(a) Re-express z_1 and z_2 in correct modulus-argument form with θ in the interval $-\pi < \theta \leq \pi$.

[3]

(b) Work out

(i) $z_1 z_2$

(ii) $\frac{z_2}{z_1}$

giving your answers in modulus-argument form with θ in the interval $-\pi < \theta \leq \pi$.

[4]

(c) (i) On an Argand diagram, show the complex numbers z_1, z_2 and $z_1 + z_2$ as position vectors.

(ii) Use your diagram to explain briefly why we speak of a 'parallelogram rule' for complex number addition.

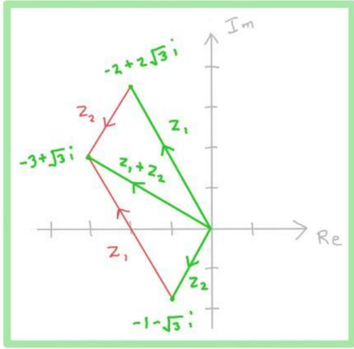
[4]

Note: The other diagonal of the parallelogram is (as a vector) equal to the difference of z_1 and z_2.

c) (i) $z_1 = 4\left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right) = -2 + 2\sqrt{3}\,i$

$z_2 = 2\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) = -1 - \sqrt{3}\,i$

$z_1 + z_2 = -3 + \sqrt{3}\,i$



(ii) The vectors in the diagram form a parallelogram, with $z_1 + z_2$ being its diagonal as shown.

Q7

On an Argand diagram, shade the region which satisfies both of the following inequalities:

$$|z - 4\mathrm{i}| > |z| \quad \text{and} \quad |z - 4\mathrm{i}| < |z + 2|$$

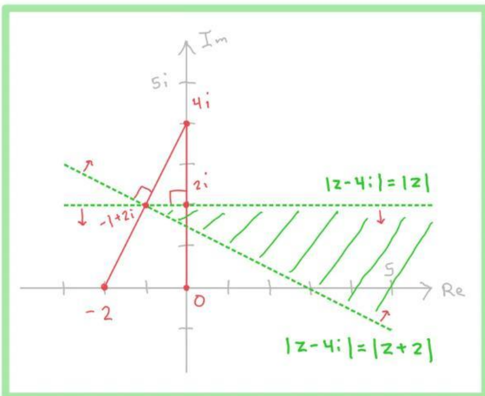
[5]

All the numbers z that are closer to 0 than they are to 4i in an Argand diagram

All the numbers z that are closer to 4i than they are to −2 in an Argand diagram

The 'equals' lines in the diagram are the perpendicular bisectors of the line segments between the relevant points.

$|z+2| = |z-(-2)|$



Note: 'greater than' and 'less than' mean that the green 'equals' lines are not included in the shaded area here.

Q8

On an Argand diagram, shade the region which satisfies all three of the following inequalities:

$|z + 4 - 3i| \le 3$ $\quad |z - (-4+3i)|$

$\frac{3\pi}{4} \le \arg(z + 1 - 3i) \le \pi$ $\quad \arg(z - (-1+3i))$

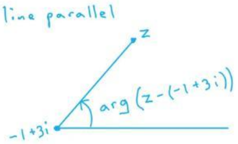
$|z + 8| > |z + 4|$ $\quad |z - (-8)| > |z - (-4)|$

[7]

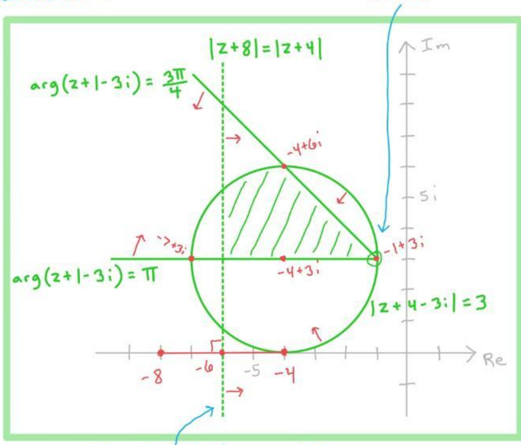
$|z+4-3i| = 3$ is a circle, radius 3, centre at $-4+3i$

All the numbers z that are closer to -4 than they are to -8 in an Argand diagram

$\arg(z-(-1+3i))$ is the angle that the line from $-1+3i$ to z makes with the line that starts at $-1+3i$ and goes off to the right parallel with the real axis. It is measured anticlockwise from the line parallel to the real axis.



$-1+3i$ is not included in the shaded area



'Greater than' means that numbers on the green dashed line are not included in the shaded area.

Q9A

a) Because $(z+(5-i))$ is a factor, $-5+i$ is a root.

The other roots are the solutions to:

$$z^2 - 6iz - 30 = 0$$

$$(z-3i)^2 + 9 - 30 = 0$$

$$(z-3i)^2 = 21$$

$$z - 3i = \pm\sqrt{21}$$

$$z = 3i \pm \sqrt{21}$$

You could also use the quadratic formula here

So the three roots are

$$\boxed{-5+i\,,\quad \sqrt{21}+3i\,,\quad -\sqrt{21}+3i}$$

Q9b

b)

$|(-5+i)-i| = |-5| = 5$

$|z| = |a+bi| = \sqrt{a^2+b^2}$

$|(\sqrt{21}+3i)-i| = |\sqrt{21}+2i|$

$= \sqrt{(\sqrt{21})^2+2^2} = \sqrt{25} = 5$

$|(-\sqrt{21}+3i)-i| = |-\sqrt{21}+2i|$

$= \sqrt{(-\sqrt{21})^2+2^2} = \sqrt{25} = 5$

$c = 5$

Q9C

$|z-i| = 5$ is a circle, radius 5, centre at i.

c)